Grade 11/12 Math Circles

March 20, 2024

Primality Testing and Integer Factorization - Problem Solutions

1. Determine whether 161 is prime, and if not, factor it.

Solution: If we trial divide by the first few primes, we notice that $161 = 7 \times 23$, which is composite.

2. Calculate the prime factorization of 1001.

Solution: If we trial divide by the first few primes, we notice that $1001 = 7 \times 11 \times 13$.

3. Prove that if a divides n and $\sqrt{n} \leq a < n$, then there exists b which divides n and satisfies $1 < b \leq \sqrt{n}$.

Solution: Let $b = n/a$. Since $a < n$, $b > 1$, and since $a \geq \sqrt{n}$, $b \leq n/\sqrt{n} = \sqrt{n}$.

4. Determine whether 1739 and 1741 are prime, and if not, factor them.

Solution: We only need to trial divide by the primes up to $\sqrt{1741} \approx 42$. There are only 13 of them, so it's not too bad to try them all. We get that $1739 = 37 \times 47$, and 1741 is prime.

5. Find the prime factorization of 344929.

Solution: We must trial divide by the primes up to $\sqrt{344929} \approx 587$. This looks daunting, but we quickly notice that $344929 = 13 \times 26533$, then that $26533 = 13 \times 2041$, then that $2014 = 13 \times 157$. Finally, since $157 < 13^2$, it must be prime, so the desired factorization is $344929 = 13^3 \times 157$.

6. Using the prime number theorem, approximately how many primes are less than 100?

Solution: We have $\pi(100) \approx \frac{100}{\ln(100)} \approx 22$. (The actual number is 25).

7. Suggest an algorithm to calculate the primes between two positive real numbers x and y (for example, $x = 100$ and $y = 100$). Notice that the Sieve of Eratosthenes would not work without modification, since 2 would never be detected as a prime and thus even numbers would not be struck out.

Solution: The Sieve of Eratosthenes works for $x = 0$ because it calculates all primes up to $\sqrt{y}$ as part of the process and sieves out their multiples. If $x \neq 0$, we could use a regular sieve to get the list of primes up to $\sqrt{y}$, then strike out just their multiples in the range x to y. This will indeed produce the list of primes between x and y, since every composite number $\leq y$ has a prime factor $\leq \sqrt{y}$. (This is an open-ended question and other algorithms may work).

8. Find a factor of 999991.

Solution: Notice that $999991 = 1000^2 - 3^2 = (1000 - 3)(1000 + 3) = 997 \times 1003$.

9. (Challenge) Find a factor of 2146681.

Solution: Notice that $2146681 + 2^3 = 129^3$. We have the identity $x^3 - y^3 = (x - y)(x^2 + xy + y^2)$, upon which we obtain the factor $129 - 2 = 127$.